

# Geodetic precession and strong gravitational lensing in the dynamical Chern-Simons modified gravity

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## Abstract

We have investigated the geodetic precession and the strong gravitational lensing in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity theory. We present the formulas of the orbital period  $T$  and the geodetic precession angle  $\Delta\Theta$  for the timelike particles in the circular orbits around the black hole, which shows that the change of the geodetic precession angle with the Chern-Simons coupling parameter  $\xi$  is converse to the change of the orbital period with  $\xi$  for fixed  $a$ . We also discuss the effects of the Chern-Simons coupling parameter on the strong gravitational lensing when the light rays pass close to the black hole and obtain that for the stronger Chern-Simons coupling the prograde photons may be captured more easily, and conversely, the retrograde photons is harder to be captured in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. Supposing that the gravitational field of the supermassive central object of the Galaxy can be described by this metric, we estimated the numerical values of the main observables for gravitational lensing in the strong field limit.

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## I. INTRODUCTION

Since Einstein's general relativity was set forth in the last century, it has been of interest in the study of possible modifications of his theory. One of the most promising extension of general relativity is Chern-Simons modified gravity [1–3], in which the Einstein-Hilbert action is modified by adding a parity-violating Chern-Simons term, which couples to gravity via a scalar field. The parity-violating Chern-Simons term is a combination of the second order in the curvature tensor and a Chern-Simons scalar field. Thus, the Chern-Simons modified gravity is a high-energy extension of Einstein's general relativity. Actually, Chern-Simons correction is necessary in the string theory as an anomaly-canceling term to conserve unitarity [4–7]. In loop quantum gravity, it is required to ensure gauge invariance of the Ashtekar variables [8]. Moreover, the Chern-Simons modified gravity could help us to explain several problems in cosmology, such as, dark energy and dark matter [9], baryon asymmetry [4, 10, 11], and so on.

It is well known that there exist two formulations of Chern-Simons modified gravity. In the non-dynamical formulation, the Chern-Simons scalar field is an *a priori* prescribed function so that its effective evolution equation can be reduced to a differential constraint on the space of allowed solutions [12–14]. While in the dynamical formulation, the Chern-Simons scalar field is treated as a dynamical field possessing its own stress-energy tensor and an evolution equation [15–17]. It must be pointed that although the non-dynamical Chern-Simons gravity action can be obtained as a certain limit of the dynamical Chern-Simons gravity action, the non-dynamical Chern-Simons gravity and dynamical Chern-Simons gravity are inequivalent and independent theories. In general, the solutions of the non-dynamical Chern-Simons gravity cannot be obtained from the solutions of the dynamical Chern-Simons gravity [16].

The characteristic observational signature of the Chern-Simons modified gravity could allow us to discriminate an effect of this theory from other phenomena. Since the Schwarzschild solution is unaffected by Chern-Simons modified gravity, the solar system tests of general relativity do not put strict bounds on the magnitude of this correction. Recently, Cardoso *et al.* [18] studied the evolution of the dynamical Chern-Simons perturbations in the background of a Schwarzschild black hole and found that the quasinormal modes could offer a way to detect the correction from dynamical Chern-Simons terms. For a rotating black hole, it is allowed to possess a non-vanishing Chern-Simons scalar field, which brings convenient for us to probe the observational signature of the Chern-Simons modified gravity. Thus, a lot of attention have been focused on the studies of rotating black holes in the Chern-Simons modified gravity [9, 13, 14, 16, 17, 19–25]. In the

non-dynamical formulation, Alexander and Yunes [13, 14] adopted to a far-field approximation and obtained a rotating black hole solution in the Chern-Simons modified gravity. In the non-dynamical framework, it was found that the Chern-Simons modified theory predicts an anomalous precession effect [14], which was tested with LAGEOS [15, 19, 20]. Using double binary pulsar data, Yunes and Spergel [21] obtained a bound on the non-dynamical model with a canonical Chern-Simons scalar field. In the dynamical Chern-Simons modified gravity, Yunes and Pretorius [16] found a rotating black hole solution by using of the small-coupling and slow-rotation limit. This rotating black hole solution was also obtained by Konno *et al.* in [17]. Harko [22] studied the properties of the thin accretion disk around this black hole and probed the effect of the Chern-Simons coupling parameter on the flux and the emission spectrum of the accretion disks. Amarilla *et al.* [23] studied the null geodesics corresponding to a slowly-rotating black hole in the dynamical Chern-Simons gravity and discussed the effect of the Chern-Simons term on the shadows casted by a black hole. These results are very useful for us to understand the properties of black holes in the Chern-Simons modified gravity. The main purpose of this paper is to study geodetic precession and the strong gravitational lensing in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity theory and to see what effects of the Chern-Simons coupling parameter on the geodetic precession angle for the timelike particles and the coefficients of gravitational lensing in the strong field limit.

The plan of our paper is organized as follows. In Sec.II we introduce briefly the dynamical formulation of the Chern-Simons modified gravity and present the a slowly-rotating black hole solution found in [16, 17]. In Sec.III, we calculate the effects of the Chern-Simons term on the geodetic precession in the circular orbits around this black hole. In Sec.IV, we adopt to Bozza's method [26, 27] and obtain the deflection angles for light rays passing close to the slowly-rotating black hole in the dynamical Chern-Simons modified gravity, and then probe the effects of the Chern-Simons coupling parameter on the deflection angle and on the coefficients in the strong field limit. At last, we present a summary.

## II. SLOWLY-ROTATING BLACK HOLE IN THE DYNAMICAL CHERN-SIMONS MODIFIED GRAVITY

In this section, we introduce briefly the slowly-rotating black hole in the dynamical Chern-Simons modified gravity [16, 17]. The action in the dynamical Chern-Simons modified gravity theory can be expressed as

$$S = \int d^4x \sqrt{-g} \left[ \kappa R + \frac{\alpha}{4} \vartheta^* R R - \frac{\beta}{2} \left( g^{\mu\nu} \nabla_\mu \vartheta \nabla_\nu \vartheta + V(\vartheta) \right) + \mathcal{L}_{\text{matt}} \right]. \quad (1)$$

The first term in the right side of Eq.(1) is the standard Einstein-Hilbert term with  $\kappa^{-1} = 16\pi G$ . The second term denotes the Chern-Simons correction, which consists of the product of a Chern-Simons scalar field  $\vartheta$  and the Pontryagin density  $*RR$ , defined via  $*RR = *R^a{}_{b}{}^{cd}R^b{}_{acd}$ . The dual Riemann-tensor  $*R^a{}_{b}{}^{cd}$  is given by

$$*R^a{}_{b}{}^{cd} = \frac{1}{2}\epsilon^{cdef}R^a{}_{bef}, \quad (2)$$

with  $\epsilon^{cdef}$  is the 4-dimensional Levi-Civita tensor. The parameters  $\alpha$  and  $\beta$  are dimensional coupling constants. The coupling constant  $\beta$  is allowed to be arbitrary in the dynamical formulation of the Chern-Simons modified gravity, but it is set to zero in the non-dynamical framework [16].

Varying the action  $S$  with respect to the metric and the Chern-Simons coupling field, one can find that the modified gravitational field equation and the motion equation of the scalar field  $\vartheta$  obey

$$G_{\mu\nu} + \frac{\alpha}{\kappa}C_{\mu\nu} = \frac{1}{2\kappa}(T_{\mu\nu}^{matt} + T_{\mu\nu}^{\vartheta}), \quad (3)$$

and

$$\beta\nabla_{\mu}\nabla^{\mu}\vartheta = \beta\frac{dV(\vartheta)}{d\vartheta} - \frac{\alpha}{4}*RR, \quad (4)$$

respectively. Here  $G_{\mu\nu}$  is the Einstein tensor and  $C_{\mu\nu}$  is the Cotton tensor. Obviously, the evolution of the scalar field  $\vartheta$  depends not only on its stress-energy tensor, but also on the curvature of spacetime.

Employing the small-coupling and slow-rotating approximation, one can obtain a black-hole solution with non-zero coupling constants in the dynamical Chern-Simons modified gravity, which can be expressed as [16, 17]

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 - 2g_{t\phi}dtd\phi, \quad (5)$$

$$\vartheta = \frac{5}{8}\frac{\alpha}{\beta}\frac{a}{M}\frac{\cos\theta}{r^2}\left(1 + \frac{2M}{r} + \frac{18M^2}{5r^2}\right), \quad (6)$$

with

$$\begin{aligned} g_{tt} &= 1 - \frac{2M}{r} + \frac{2a^2M}{r^3}\cos^2\theta, \\ g_{rr} &= \left(1 - \frac{2M}{r}\right)^{-1}\left[1 + \frac{a^2}{r}\left(\cos^2\theta - \left(1 - \frac{2M}{r}\right)^{-1}\right)\right], \\ g_{\theta\theta} &= r^2 + a^2\cos^2\theta, \\ g_{t\phi} &= \frac{2Ma}{r}\sin^2\theta - \frac{5\xi a}{8r^4}\left(1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right)\sin^2\theta, \\ g_{\phi\phi} &= r^2\sin^2\theta + a^2\sin^2\theta\left(1 + \frac{2M}{r}\sin^2\theta\right). \end{aligned} \quad (7)$$

Here the parameter  $\xi$  is related to the coupling constants  $\alpha$  and  $\beta$  by  $\xi = \alpha^2/(\beta\kappa)$ , which has an exact dimension  $[L]^4$ . As in ref.[16], we can define a dimensionless parameter  $\zeta$  by re-scaling by a factor  $(2M)^4$ , i.e.,  $\zeta = \xi/(2M)^4$ . If  $\alpha$  tends to zero, one can find that Chern-Simons scalar field  $\vartheta$  disappears and the metric (5) return to that of the slow-rotating Kerr black hole in the general relativity. Since the Chern-Simons scalar field  $\vartheta$  has positive energy [16], it is natural that the parameter  $\xi$  is non-negative. Here we limit ourselves to the case where  $\xi \geq 0$  and study the effect of  $\xi$  on the geodetic precession and the strong gravitational lensing in the background (5).

### III. GEODETIC PRECESSION IN THE SLOWLY-ROTATING BLACK HOLE IN THE DYNAMICAL CHERN-SIMONS MODIFIED GRAVITY

The timelike geodesics in the slowly-rotating black hole in dynamical Chern-Simons modified gravity were considered in [22, 25]. Harko [22] studied the properties of the thin accretion disk around the black hole (5). Sopuerta *et al*[25] considered the timelike geodesic equations for the massive particles and found that the location of the innermost stable circular orbit (ISCO) and the three physical fundamental frequencies associated with the time  $\tau$  for the particles are modified in the Chern-Simons modified gravity. However, in Ref.[25] the geodesic precession of orbits around Chern-Simons black holes is only illustrated numerically for a few examples, while an analytic expression for this physical quantity is still missing. In this paper, we will present an clear expression of  $T$  and study the effects of the Chern-Simons term on the Kepler's third law, and then study the geodetic precession of the massive particles around the black hole (5).

Let us start with the condition  $\theta = \pi/2$ , which set the orbits on the equatorial plane. In this case, one can find that the timelike geodesics take the form

$$u^t = \frac{dt}{d\tau} = \frac{Eg_{\phi\phi} - Lg_{t\phi}}{g_{t\phi}^2 + g_{tt}g_{\phi\phi}}, \quad (8)$$

$$u^\phi = \frac{d\phi}{d\tau} = \frac{Eg_{t\phi} + Lg_{tt}}{g_{t\phi}^2 + g_{tt}g_{\phi\phi}}, \quad (9)$$

$$\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) = E^2, \quad (10)$$

with the effective potential

$$V_{eff}(r) = \frac{1}{g_{rr}} \left( 1 + \frac{E^2[g_{rr}(g_{t\phi}^2 + g_{tt}g_{\phi\phi}) - g_{\phi\phi}] + 2ELg_{t\phi} + L^2g_{tt}}{g_{t\phi}^2 + g_{tt}g_{\phi\phi}} \right), \quad (11)$$

where  $E$  and  $L$  are the specific energy and the specific angular momentum of particles moving in the orbits, respectively. For the stable circular orbit in the equatorial plane, the effective potential  $V(r)$  must obey

$$V_{eff}(r) = E^2, \quad \frac{dV_{eff}(r)}{dr} = 0. \quad (12)$$

Solving above equations, one can obtain

$$\begin{aligned} E &= \frac{g_{tt} + g_{t\phi}\Omega}{\sqrt{g_{tt} + 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \\ L &= \frac{-g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{g_{tt} + 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \\ \Omega &= \frac{d\phi}{dt} = \frac{g_{t\phi,r} + \sqrt{(g_{t\phi,r})^2 + g_{tt,r}g_{\phi\phi,r}}}{g_{\phi\phi,r}}, \end{aligned} \quad (13)$$

where  $\Omega$  is the angular velocity of particle moving in the orbits. From Eq.(13), one can obtain Kepler's third law in the slowly-rotating black-hole spacetime in the dynamical Chern-Simons modified gravity

$$\begin{aligned} T^2 &= \frac{4\pi^2}{M} R^3 \left[ 1 + a \frac{112MR^5 - \xi(567M^2 + 300MR + 140R^2)}{56M^{1/2}R^{13/2}} \right. \\ &\quad \left. + \frac{a^2 M}{R^3} \left( 1 - \xi \frac{567M^2 + 300MR + 140R^2}{28MR^5} + \xi^2 \frac{(567M^2 + 300MR + 140R^2)^2}{6272M^2R^{10}} \right) + \mathcal{O}(a^3) \right], \end{aligned} \quad (14)$$

where  $T$  is the orbital period and  $R$  is the radius of the circular orbit. The later terms in the right hand side is the correction by the  $a$  and the Chern-Simons term. Obviously, the correction term disappears as  $a$  approaches zero. It is reasonable because that as  $a$  vanishes the metric (5) reduces to that of the Schwarzschild black hole in the general relativity. Since the black hole is slowly rotating, the correction is dominated by the first-order terms in  $a$ . Thus, when the black hole rotates in the same direction as the particle, i.e.,  $a > 0$ , the orbital period  $T$  decreases with the Chern-Simons coupling parameter  $\xi$ . But when the black hole rotates in the converse direction as the particle, i.e.,  $a < 0$ , the orbital period  $T$  increases with the Chern-Simons coupling parameter  $\xi$ .

Now, we are in the position to study the geodetic precession of a timelike particle in the circular orbits around the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. As in [29], we regard the rotating axis of the gyroscope carried by a satellite as a spacelike spin-vector  $S^\mu$ , which parallelly transported along a timelike geodesic with the four-velocity  $u^\mu$ . Thus, the parallel transporting equation of  $S^\mu$  in the direction of  $u^\mu$  can be expressed as

$$u^\mu \nabla_\mu S^\nu = 0. \quad (15)$$

From the orthogonality and normalization conditions, one can find

$$u^\mu S_\mu = 0, \quad S^\mu S_\mu = 1. \quad (16)$$

In the slowly-rotating black hole in the dynamical Chern-Simons modified gravity, the parallel transporting

equation (15) along the circular orbits in the equatorial plane reads

$$\frac{dS^t}{d\tau} + \mathcal{A}S^r = 0, \quad (17)$$

$$\frac{dS^r}{d\tau} + \mathcal{B}S^t + \mathcal{C}S^\phi = 0, \quad (18)$$

$$\frac{dS^\theta}{d\tau} = 0, \quad (19)$$

$$\frac{dS^\phi}{d\tau} + \mathcal{D}S^r = 0, \quad (20)$$

with

$$\mathcal{A} = \frac{1}{2} \left[ \left( \frac{g_{tt,r}g_{\phi\phi} + g_{t\phi,r}g_{t\phi}}{g_{t\phi}^2 + g_{tt}g_{\phi\phi}} \right) u^t + \left( \frac{g_{t\phi,r}g_{\phi\phi} - g_{t\phi}g_{\phi\phi,r}}{g_{t\phi}^2 + g_{tt}g_{\phi\phi}} \right) u^\phi \right] \Big|_{r=R}, \quad (21)$$

$$\mathcal{B} = \frac{1}{2} \left[ \left( \frac{g_{tt,r}}{g_{rr}} \right) u^t + \left( \frac{g_{t\phi,r}}{g_{rr}} \right) u^\phi \right] \Big|_{r=R}, \quad (22)$$

$$\mathcal{C} = \frac{1}{2} \left[ \left( \frac{g_{t\phi,r}}{g_{rr}} \right) u^t - \left( \frac{g_{\phi\phi,r}}{g_{rr}} \right) u^\phi \right] \Big|_{r=R}, \quad (23)$$

$$\mathcal{D} = \frac{1}{2} \left[ \left( \frac{g_{tt,r}g_{t\phi} - g_{tt}g_{t\phi,r}}{g_{t\phi}^2 + g_{tt}g_{\phi\phi}} \right) u^t + \left( \frac{g_{t\phi,r}g_{t\phi} + g_{tt}g_{\phi\phi,r}}{g_{t\phi}^2 + g_{tt}g_{\phi\phi}} \right) u^\phi \right] \Big|_{r=R}. \quad (24)$$

Combining above equations with Eqs. (8) and (9), we can obtain the spin-vector  $S^\mu$

$$S^t = C^t \sin(\varpi\tau), \quad (25)$$

$$S^r = C^r \cos(\varpi\tau), \quad (26)$$

$$S^\theta = C^\theta, \quad (27)$$

$$S^\phi = C^\phi \sin(\varpi\tau), \quad (28)$$

with

$$\varpi = \sqrt{-(\mathcal{A}\mathcal{B} + \mathcal{C}\mathcal{D})}. \quad (29)$$

Here we have imposed initial condition  $S^t = S^\phi = 0$  at  $\tau = 0$ . The coefficients  $C^t$ ,  $C^r$ ,  $C^\theta$  and  $C^\phi$  can be constrained by the orthogonality and normalization conditions (16).

As  $\phi$  goes from 0 to  $2\pi$ , one can obtain that the proper time  $\tau$  goes from 0 to  $\tau_p = 2\pi/u^\phi$ . Thus, the geodetic precession angle  $\Delta\Theta$  during one orbital period can be expressed as

$$\Delta\Theta = |\varpi\tau_p - 2\pi| = \left| 2\pi \left( \frac{\varpi}{u^\phi} - 1 \right) \right|. \quad (30)$$

Substituting (9) and (29) into (30), one can expand Eq. (30) as a power series in  $M/R$  up to  $\mathcal{O}(\frac{M^5}{R^5})$  and obtain the geodetic precession angle  $\Delta\Theta$  in the slowly-rotating black hole in dynamical Chern-Simons modified

gravity

$$\Delta\Theta = \frac{3\pi M}{R} \left[ \left( 1 + \frac{3M}{4R} + \frac{9M^2}{8R^2} + \frac{135M^3}{64R^3} \right) - \frac{a}{\sqrt{MR}} \left( \frac{2}{3} + \frac{M}{R} + \frac{9M^2}{4R^2} + \frac{45M^3}{8R^3} - \frac{5\xi}{6MR^3} \right) + \frac{a^2}{R^2} \left( \frac{1}{3} + \frac{3M}{2R} + \frac{45M^2}{8R^2} \right) + \mathcal{O}(a^3) \right]. \quad (31)$$

From Eq. (31), it is easy to find that the geodetic precession angle  $\Delta\Theta$  increases with the Chern-Simons coupling parameter  $\xi$  if  $a > 0$  and it decreases with  $\xi$  if  $a < 0$ . Comparing with Eq. (14), we find that the dependent of the geodetic precession angle on the  $\xi$  is converse to the dependent of the orbital period on the  $\xi$ . It is understandable by a fact that the increase of the orbital period  $T$  leads to the decrease of the angular velocity  $\omega$  of particle and then it results in the decrease of the precession angle  $\Delta\Theta$ .

#### IV. DEFLECTION ANGLE IN THE SLOWLY-ROTATING BLACK HOLE IN THE DYNAMICAL CHERN-SIMONS MODIFIED GRAVITY

The null geodesics was considered to study the properties of the shadows casted by a slowly-rotating black hole in the dynamical Chern-Simons gravity [23]. In this section, we will study deflection angles of the light rays when they pass close to the slowly-rotating black hole in dynamical Chern-Simons modified gravity, and then probe the effects of the Chern-Simons coupling parameter  $\xi$  on the deflection angle and the coefficients in the strong field limit.

##### A. Formulas in the strong gravitational lensing

As in the former, we also consider only the case the light ray is limited in the equatorial plane. With this condition, the reduced metric for the slowly-rotating black hole in dynamical Chern-Simons modified gravity can be expressed as

$$ds^2 = -A(x)dt^2 + B(x)dx^2 + C(x)d\phi^2 - 2D(x)dt d\phi, \quad (32)$$

where we adopt to a new radial coordinate  $x = r/2M$  and the metric coefficients have the form

$$A(x) = 1 - \frac{1}{x}, \quad (33)$$

$$B(x) = \frac{x^2 - x - \hat{a}^2}{(x-1)^2}, \quad (34)$$

$$C(x) = x^2 + \frac{\hat{a}^2(x+1)}{x}, \quad (35)$$

$$D(x) = \frac{\hat{a}}{x} - \frac{\hat{a}\zeta(280x^2 + 189x + 240)}{448x^6}. \quad (36)$$



Here the quantities  $\zeta = \frac{\xi}{(2M)^4}$  and  $\hat{a} = \frac{a}{2M}$  are the re-scaled Chern-Simons coupling parameter and the re-scaled rotation parameter of black hole, respectively. Obviously, the parameters  $\zeta$  and  $\hat{a}$  are dimensionless. For simplicity, we set  $2M = 1$  in the following calculations.

As in Ref. [27], the null geodesics take the form

$$\frac{dt}{d\lambda} = \frac{C(x) - JD(x)}{D(x)^2 + A(x)C(x)}, \quad (37)$$

$$\frac{d\phi}{d\lambda} = \frac{D(x) + JA(x)}{D(x)^2 + A(x)C(x)}, \quad (38)$$

where  $\lambda$  is an affine parameter along the geodesics and  $J$  is the angular momentum of the photon.

For the null geodesics, the Lagrangian  $\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$  vanishes. This implies that

$$\dot{x} = \pm \sqrt{\frac{C(x) - J[2D(x) + JA(x)]}{B(x)[D(x)^2 + A(x)C(x)]}}. \quad (39)$$

Clearly,  $\dot{x}$  is equal to zero at the minimum distance of approach of the light ray. Combining with Eqs. (37) and (38), one can obtain that [27]

$$J = u = \frac{-D(x_0) + \sqrt{A(x_0)C(x_0) + D^2(x_0)}}{A(x_0)}. \quad (40)$$

where  $x_0$  is the closest approach distance and  $u$  is the impact parameter. In the slowly-rotating black-hole spacetime in the dynamical Chern-Simons modified gravity, the photon-sphere equation is given by

$$A(x)C'(x) - A'(x)C(x) + 2J[A'(x)D(x) - A(x)D'(x)] = 0. \quad (41)$$

Obviously, this equation is more complex than that in the background of a static and spherical black hole [30]. It is difficult to obtain an analytical form for the photon-sphere radius in this case. However, we can expand Eq.(41) as a power series in  $\hat{a}$  and find that the photon-sphere radius in the slowly-rotating approximation can be expressed as

$$x_{ps} = \frac{3}{2} - \hat{a} \left( \frac{2\sqrt{3}}{3} - \frac{62\sqrt{3}}{243}\zeta \right) - \hat{a}^2 \left( \frac{4}{9} - \frac{17924}{5103}\zeta + \frac{896024}{413343}\zeta^2 \right) + \mathcal{O}(\hat{a}^3). \quad (42)$$

From Eq.(42), one can obtain that the photon-sphere radius  $x_{ps}$  increases with the parameter  $\zeta$  if the photons are winding in the same direction of the black-hole rotation (i.e.,  $\hat{a} > 0$ ), while the radius  $x_{ps}$  decreases with the parameter  $\zeta$  if the photons rotate in converse direction to the black hole (i.e.,  $\hat{a} < 0$ ). This is also shown in Fig.(1) in which we plotted the variety of the photon-sphere radius  $x_{ps}$  with the Chern-Simons coupling parameter  $\zeta$  and the rotating parameter  $\hat{a}$  by solving Eq. (41) numerically. Moreover, we find that as  $\zeta \rightarrow 0$ , the photon-sphere radius  $x_{ps}$  reduces to that in Kerr black hole. As the rotation parameter  $\hat{a}$  tends to zero,  $x_{ps}$  is independent of the Chern-Simons coupling parameter  $\zeta$ .

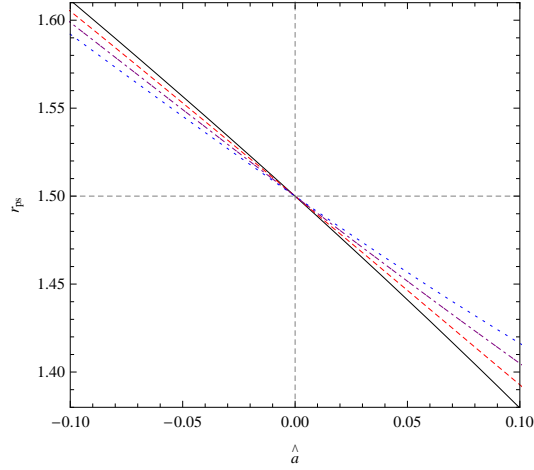


FIG. 1: Variety of the quantity  $r_{ps} = 2Mx_{ps}$  with the Chern-Simons coupling parameter  $\zeta$  in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. The solid, dashed, dash-dotted and dotted curves are for  $\zeta = 0, 0.1, 0.2$  and  $0.3$ , respectively. Here we set  $2M = 1$ .

Following ref. [31], we can obtain the deflection angle for the photon coming from infinite in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity.

$$\alpha(x_0) = I(x_0) - \pi, \quad (43)$$

and  $I(x_0)$  is

$$I(x_0) = 2 \int_{x_0}^{\infty} \frac{\sqrt{B(x)|A(x_0)|[D(x) + JA(x)]dx}}{\sqrt{D^2(x) + A(x)C(x)}\sqrt{A(x_0)C(x) - A(x)C(x_0) + 2J[A(x)D(x_0) - A(x_0)D(x)]}}, \quad (44)$$

As in the Schwarzschild black hole spacetime, the deflection angle increases when parameter  $x_0$  decreases. For a certain value of  $x_0$  the deflection angle becomes  $2\pi$ , so that the light ray makes a complete loop around the black hole before reaching the observer. If  $x_0$  is equal to the radius of the photon sphere  $x_{ps}$ , one can find that the deflection angle diverges and the photon is captured on a circular orbit. From the above discussion about the variety of the photon-sphere radius  $r_{ps}$  with the Chern-Simons parameter  $\zeta$ , it is easy to obtain that for the larger  $\zeta$  the prograde photons may be captured more easily, and conversely, the retrograde photons is harder to be captured.

In order to find the behavior of the deflection angle very close to the photon sphere, we adopt to the evaluation method for the integral (44) proposed by Bozza [26], which has been widely used in studying of the strong gravitational lensing of various black holes [30, 32–46]. Let us now to define a variable

$$z = 1 - \frac{x_0}{x}, \quad (45)$$

and rewrite the Eq.(44) as

$$I(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz, \quad (46)$$

with

$$R(z, x_0) = 2 \frac{1 - A(x_0)}{A'(z) \sqrt{C(z)}} \frac{\sqrt{B(z)A(x_0)} [D(z) + JA(z)]}{\sqrt{D^2(z) + A(z)C(z)}}, \quad (47)$$

$$f(z, x_0) = \frac{1}{\sqrt{A(x_0) - A(z) \frac{C(x_0)}{C(z)} + \frac{2J}{C(z)} [A(z)D(x_0) - A(x_0)D(z)]}}. \quad (48)$$

The function  $R(z, x_0)$  is regular for all values of  $z$  and  $x_0$ . However, the function  $f(z, x_0)$  diverges as  $z$  tends to zero, i.e., as the photon approaches the photon sphere. Thus, we can split the integral (46) into the divergent part  $I_D(x_0)$  and the regular one  $I_R(x_0)$

$$\begin{aligned} I_D(x_0) &= \int_0^1 R(0, x_{ps}) f_0(z, x_0) dz, \\ I_R(x_0) &= \int_0^1 [R(z, x_0) f(z, x_0) - R(0, x_{ps}) f_0(z, x_0)] dz. \end{aligned} \quad (49)$$

Expanding the argument of the square root in  $f(z, x_0)$  to the second order in  $z$ , we have

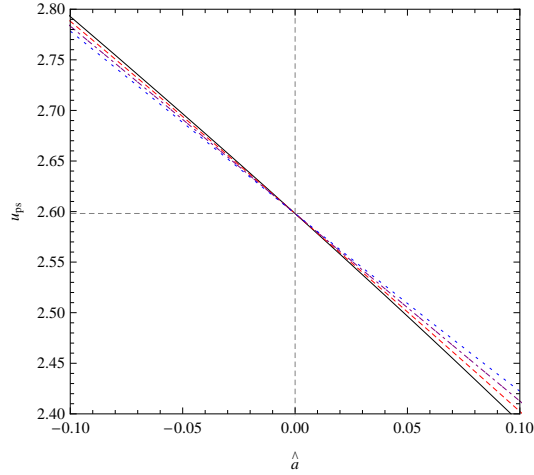


FIG. 2: The minimum impact parameter  $u_{ps}$  changes with the Chern-Simons coupling parameter  $\zeta$  in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. The solid, dashed, dash-dotted and dotted curves are for  $\zeta = 0, 0.1, 0.2$  and  $0.3$ , respectively. Here we set  $2M = 1$ .

$$f_0(z, x_0) = \frac{1}{\sqrt{p(x_0)z + q(x_0)z^2}}, \quad (50)$$

where

$$\begin{aligned}
p(x_0) &= \frac{1 - A(x_0)}{A'(x_0)C(x_0)} \left\{ A(x_0)C'(x_0) - A'(x_0)C(x_0) + 2J[A'(x_0)D(x_0) - A(x_0)D'(x_0)] \right\}, \\
q(x_0) &= \frac{(1 - A(x_0))^2}{2A'^3(x_0)C^2(x_0)} \left( 2C(x_0)C'(x_0)A'^2(x_0) + [C(x_0)C''(x_0) - 2C'^2(x_0)]A(x_0)A'(x_0) \right. \\
&\quad \left. - C(x_0)C'(x_0)A(x_0)A''(x_0) + 2J\{A(x_0)C(x_0)[A''(x_0)D'(x_0) - A'(x_0)D''(x_0)] \right. \\
&\quad \left. + 2A'(x_0)C'(x_0)[A(x_0)D'(x_0) - A'(x_0)D(x_0)] \right\}. \tag{51}
\end{aligned}$$

Comparing Eq.(41) with Eq.(51), one can find that if  $x_0$  approaches to the radius of photon sphere  $x_{ps}$  the coefficient  $p(x_0)$  vanishes and the leading term of the divergence in  $f_0(z, x_0)$  is  $z^{-1}$ . This means that the integral (46) diverges logarithmically. The coefficient  $q(x_0)$  takes the form

$$q(x_{ps}) = \frac{(1 - A(x_{ps}))^2}{2A'^2(x_{ps})C(x_{ps})} \left\{ A(x_{ps})C''(x_{ps}) - A''(x_{ps})C(x_{ps})2J[A''(x_{ps})D(x_{ps}) - A(x_{ps})D''(x_{ps})] \right\}. \tag{52}$$

Therefore the deflection angle in the strong field region can be expanded in the form [26]

$$\alpha(\theta) = -\bar{a} \log \left( \frac{u}{u_{ps}} - 1 \right) + \bar{b} + \mathcal{O}(u - u_{ps}), \tag{53}$$

with

$$\begin{aligned}
\bar{a} &= \frac{R(0, x_{ps})}{\sqrt{q(x_{ps})}}, \\
\bar{b} &= -\pi + b_R + \bar{a} \log \left\{ \frac{2q(x_{ps})C(x_{ps})}{u_{ps}A(x_{ps})[D(x_{ps}) + JA(x_{ps})]} \right\}, \\
b_R &= I_R(x_{ps}), \\
u_{ps} &= \frac{-D(x_{ps}) + \sqrt{A(x_{ps})C(x_{ps}) + D^2(x_{ps})}}{A(x_{ps})}. \tag{54}
\end{aligned}$$

Making use of Eqs.(53) and (54), we can study the properties of strong gravitational lensing in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. Neglecting terms of order  $\mathcal{O}(\hat{a}^3)$  and higher term, we can expand the the coefficients ( $\bar{a}$  and  $\bar{b}$ ) in the strong gravitational lensing and the minimum impact parameter  $u_{ps}$  as a power series in  $\hat{a}$

$$\begin{aligned}
\bar{a} &= 1 + \frac{2\sqrt{3}\hat{a}(1134 - 2003\zeta)}{5103} + \hat{a}^2 \left( \frac{10}{9} - \frac{133288\zeta}{15309} + \frac{57184906\zeta^2}{8680203} \right) + \mathcal{O}(\hat{a}^3), \\
\bar{b} &= -0.40023 - (0.190505 - 2.06119\zeta)\hat{a} - \hat{a}^2(14.903\zeta^2 - 13.9387\zeta + 0.541507) + \mathcal{O}(\hat{a}^3), \\
b_R &= \log 6 + \frac{\sqrt{3}\hat{a}}{3} \left[ \frac{4(1 + \log 6)}{3} + \frac{\zeta(3754 - 4006 \log 6)}{1701} \right] \\
&\quad + \hat{a}^2 \left[ \frac{4}{3} + \frac{10}{9} \log 6 + \zeta \left( \frac{7528}{1701} - \frac{133288 \log 6}{15309} \right) - \zeta^2 \left( \frac{67484654}{8680203} - \frac{57184906 \log 6}{8680203} \right) \right] + \mathcal{O}(\hat{a}^3), \\
u_{ps} &= \frac{3\sqrt{3}}{2} - \hat{a} \left( 2 - \frac{131\zeta}{189} \right) - \frac{\sqrt{3}\hat{a}^2}{3} \left( 1 - \frac{2948\zeta}{567} + \frac{701941\zeta^2}{321489} \right) + \mathcal{O}(\hat{a}^3). \tag{55}
\end{aligned}$$

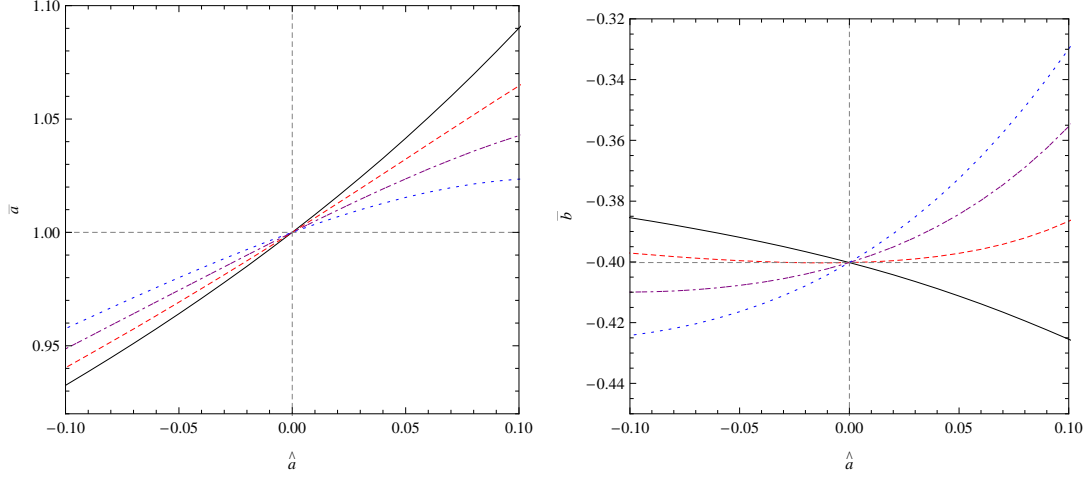


FIG. 3: Variation of the coefficients of the strong field limit  $\bar{a}$  (the left) and  $\bar{b}$  (the right) with the Chern-Simons coupling parameter  $\zeta$  in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. The solid, dashed, dash-dotted and dotted curves are for  $\zeta = 0, 0.1, 0.2$  and  $0.3$ , respectively. Here we set  $2M = 1$ .

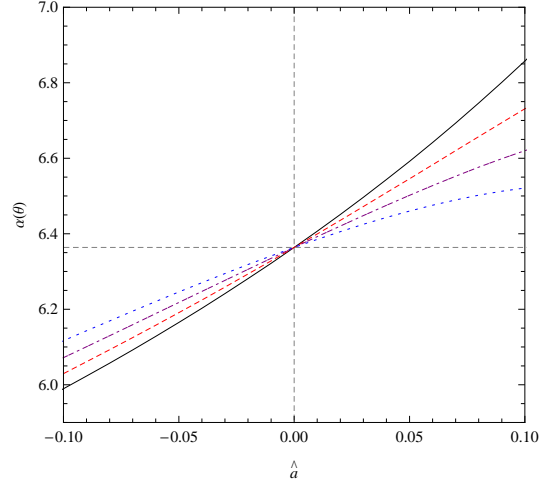


FIG. 4: Deflection angles evaluated at  $u = u_{ps} + 0.003$  is a function of the Chern-Simons coupling parameter  $\zeta$  in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. The solid, dashed, dash-dotted and dotted curves are for  $\zeta = 0, 0.1, 0.2$  and  $0.3$ , respectively. Here we set  $2M = 1$ .

In figs.(2)-(3), we plotted numerically the changes of the minimum impact parameter  $u_{ps}$  and the coefficients ( $\bar{a}$  and  $\bar{b}$ ) with  $\hat{a}$  and  $\zeta$  in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. From Eqs.(55) and figs. (2)-(3), we find that the minimum impact parameter  $u_{ps}$  and the coefficients ( $\bar{a}$  and  $\bar{b}$ ) in the strong field limit are functions of the rotation parameter  $\hat{a}$  and the Chern-Simons coupling parameter  $\zeta$ . The minimum impact parameter has similar behavior as the radius of the photon sphere  $x_{ps}$ . The coefficient  $\bar{a}$  increases with  $\hat{a}$  for fixed  $\zeta$ . For fixed  $\hat{a}$ , the coefficient  $\bar{a}$  decreases with the Chern-Simons coupling parameter  $\zeta$  if  $\hat{a} > 0$  and it increases if  $\hat{a} < 0$ . The coefficient  $\bar{b}$  decreases with  $\hat{a}$  for the smaller  $\zeta$ , but it increases with  $\hat{a}$  for the larger  $\zeta$ . For fixed  $\hat{a}$ , the variety of  $\bar{b}$  with  $\zeta$  is converse to the variety of  $\bar{a}$  with

$\zeta$ . In fig. (4), we plotted the change of the deflection angles evaluated at  $u = u_{ps} + 0.003$  with  $\zeta$ . It is shown that in the strong field limit the deflection angles have the similar properties of the coefficient  $\bar{a}$ . This means that the deflection angles of the light rays are dominated by the logarithmic term in the strong gravitational lensing. Moreover, we also find that for larger  $\zeta$  the deflection angle is larger for the retrograde photon, while it is smaller for the prograde photon. These imply that for the larger  $\zeta$  the prograde photons may be captured more easily, and conversely, the retrograde photons is harder to be captured in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity.

### B. Observables in the strong deflection limit

Let us now to study the effect of the Chern-Simons parameter  $\zeta$  on the observational gravitational lensing parameters. We start by assuming that the gravitational field of the supermassive black hole at the Galactic center of Milky Way can be described by the slowly-rotating black hole in the dynamical Chern-Simons modified gravity, and then estimate the numerical values for the main observables of gravitational lensing in the strong field limit.

In the strong deflection limit, from the lensing geometry we can rewrite the lens equation as [27]

$$\gamma = \frac{D_{OL} + D_{LS}}{D_{LS}}\theta - \alpha(\theta) \bmod 2\pi \quad (56)$$

where  $D_{LS}$  is the lens-source distance and  $D_{OL}$  is the observer-lens distance.  $\gamma$  is the angle between the direction of the source and the optical axis.  $\theta = u/D_{OL}$  is the angular separation between the lens and the image.

Following ref.[27], one can find that the angular separation between the lens and the  $n$ -th relativistic image is

$$\theta_n \simeq \theta_n^0 \left( 1 - \frac{u_{ps} e_n (D_{OL} + D_{LS})}{\bar{a} D_{OL} D_{LS}} \right), \quad (57)$$

with

$$\theta_n^0 = \frac{u_{ps}}{D_{OL}}(1 + e_n), \quad e_n = e^{\frac{\bar{b} + |\gamma| - 2\pi n}{\bar{a}}}. \quad (58)$$

The quantity  $\theta_n^0$  is the image positions corresponding to  $\alpha = 2n\pi$ , and  $n$  is an integer. According to the past oriented light ray which starts from the observer and finishes at the source the resulting images stand on the eastern side of the black hole for direct photons ( $\hat{a} > 0$ ) and are described by positive  $\gamma$ . Retrograde photons ( $\hat{a} < 0$ ) have images on the western side of the black hole and are described by negative values of  $\gamma$ .

In the limit  $n \rightarrow \infty$ , we find that  $e_n \rightarrow 0$ , and then the relation between the minimum impact parameter  $u_{ps}$  and the asymptotic position of a set of images  $\theta_\infty$  can be simplified as

$$u_{ps} = D_{OL}\theta_\infty. \quad (59)$$

In order to obtain the coefficients  $\bar{a}$  and  $\bar{b}$ , one needs to separate at least the outermost image from all the others. As in Refs.[26, 27], we consider here the simplest case in which only the outermost image  $\theta_1$  is resolved as a single image and all the remaining ones are packed together at  $\theta_\infty$ . Thus the angular separation between the first image and other ones can be expressed as [26, 27, 33]

$$s = \theta_1 - \theta_\infty = \theta_\infty e^{\frac{\bar{b}-2\pi}{\bar{a}}}. \quad (60)$$

Through measuring  $s$  and  $\theta_\infty$ , we can obtain the strong deflection limit coefficients  $\bar{a}$ ,  $\bar{b}$  and the minimum impact parameter  $u_{ps}$ . Comparing their values with those predicted by the theoretical models, we can obtain information about the parameters of the lens object stored in them.

The mass of the central object of our Galaxy is estimated recently to be  $4.4 \times 10^6 M_\odot$  [47] and its distance is around  $8.5 kpc$ , so that the ratio of the mass to the distance  $M/D_{OL} \approx 2.4734 \times 10^{-11}$ . Making use of Eqs. (55), (59) and (60) we can estimate the values of the coefficients and observables for gravitational lensing in the strong field limit. For the different  $\zeta$  and  $\hat{a}$ , the numerical value for the angular position of the relativistic images  $\theta_\infty$  and the angular separation  $s$  are listed in the Table I. The dependence of these observables on the parameters  $\zeta$  and  $\hat{a}$  are also shown in Fig. (5). Obviously, the observables  $\theta_\infty$  and  $s$  are independent of the

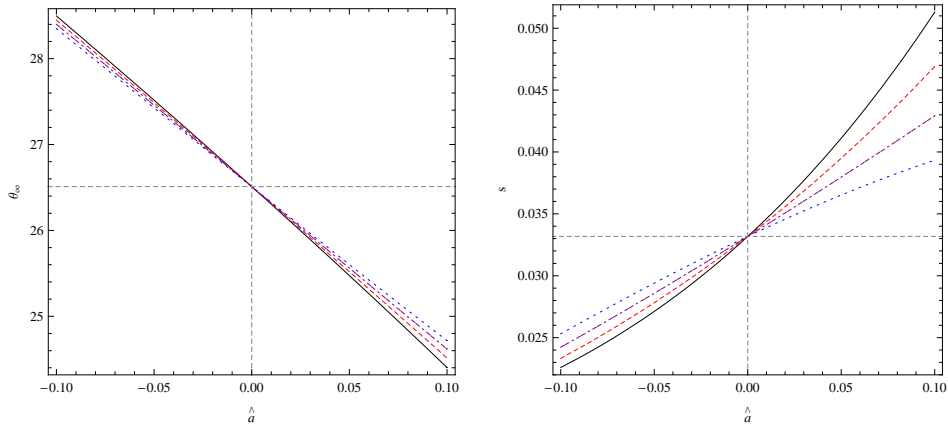


FIG. 5: Gravitational lensing by the Galactic center black hole. Variation of the values of the angular position  $\theta_\infty$ , the angular separation  $s$  with parameter  $\hat{a}$  in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity. The solid, dashed, dash-dotted and dotted curves are for  $\zeta = 0, 0.1, 0.2$  and  $0.3$ , respectively.

parameter  $\zeta$  as the rotation parameter  $\hat{a} = 0$ . From Table I and Fig. (5), we find that with the increase of

$\hat{a}$	$\theta_\infty (\mu \text{ arcsec})$				$s (\mu \text{ arcsec})$			
	$\zeta = 0$	$\zeta = 0.1$	$\zeta = 0.2$	$\zeta = 0.3$	$\zeta = 0$	$\zeta = 0.1$	$\zeta = 0.2$	$\zeta = 0.3$
-0.10	28.497	28.449	28.400	28.350	0.0226	0.0233	0.0242	0.0253
-0.05	27.516	27.487	27.458	27.428	0.0271	0.0278	0.0286	0.0294
0	26.510	26.510	26.510	26.510	0.0332	0.0332	0.0332	0.0332
0.05	25.474	25.518	25.561	25.603	0.0411	0.0395	0.0380	0.0365
0.10	24.403	24.514	24.618	24.717	0.0513	0.0469	0.0429	0.0393

TABLE I: Numerical estimation for main observables in the strong field limit for the black hole at the center of our Galaxy, which is supposed to be described by in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity.

$\zeta$ , the angular position of the relativistic images  $\theta_\infty$  increases for the direct photons ( $\hat{a} > 0$ ) and decrease for the retrograde photons ( $\hat{a} < 0$ ). The change of the angular separation  $s$  with  $\zeta$  is converse to that of  $\theta_\infty$ .

Theoretically, we could detect the effects of parameter  $\zeta$  on the strong gravitational lensing through the astronomical observations and then make a constraint on the parameter  $\xi$ . From Table I, we find that the values for  $\theta_\infty$  is very small, which leads to constrain the parameter  $\xi$  more difficultly. If one can constrain  $\zeta < 0.1$  with lensing observations, it is easy to obtain that the bound of the Chern-Simons coupling parameter is

$$\xi^{1/4} = \zeta^{1/4} 2M < 8.70 \times 10^6 \text{km}, \quad (61)$$

which is not stronger than that from the binary pulsar PSR J0737-3039 A/B [48] obtained by Yunes *et al* [16]

$$\xi^{1/4} < 1.5 \times 10^4 \text{km}. \quad (62)$$

In order that the strong gravitational lensing bound can beat the binary pulsar one, we would have to constrain  $\zeta \sim 8.44 \times 10^{-13}$ , which seems impossible in the near future.

## V. SUMMARY

In this paper we have extensively studied the geodetic precession and the strong gravitational lensing in the slowly-rotating black hole in the dynamical Chern-Simons modified gravity theory. We present the formulas of the orbital period  $T$  and the geodetic precession angle  $\Delta\Theta$  for the timelike particles in the circular orbits around the black hole. When the black hole rotates in the same direction as the particle ( $a > 0$ ), the orbital period  $T$  decreases with the Chern-Simons coupling parameter  $\xi$ . While the black hole rotates in the converse direction as the particle ( $a < 0$ ), the orbital period  $T$  increases with the parameter  $\xi$ . Moreover, it is shown that the change of the geodetic precession angle with the  $\xi$  is converse to the change of the orbital period with the  $\xi$ . We also discuss the effects of the Chern-Simons coupling parameter on the strong gravitational lensing



when the light rays pass close to the black hole. We find that the photon-sphere radius, the minimum impact parameter, and the coefficients in the strong field limit depend on the Chern-Simons coupling parameter. With the increase of  $\zeta$  (i.e., the re-scaling Chern-Simons coupling parameter  $\zeta = \xi/(2M)^4$ ) the deflection angle increases for the prograde photon, while it decreases for the retrograde photon. It means that for the larger  $\zeta$  the prograde photons may be captured more easily, and conversely, the retrograde photons is harder to be captured in the slowly-rotating black hole in dynamical Chern-Simons modified gravity.

The model was applied to the supermassive black hole in the Galactic center. Our results show that with the increase of the parameter  $\zeta$  the angular position of the relativistic images  $\theta_\infty$  increases for the direct photons ( $a > 0$ ) and decrease for the retrograde photons ( $a < 0$ ). The change of the angular separation  $s$  with  $\zeta$  is converse to that of  $\theta_\infty$ . Our result also show that the bound of  $\xi$  from the strong gravitational lensing is not stronger than that from the binary pulsar PSR J0737-3039 A/B [48]. In particular, for the case  $\zeta = 0.1$ , we can get the Chern-Simons coupling parameter  $\xi \sim 8.70 \times 10^6$  km, which is ruled out by binary pulsar constraints. If one were to choose the parameter  $\xi$  small enough not to be ruled out by these constraints, the effect would be much smaller than microarcseconds. Perhaps with the development of technology, the effects of parameter  $\zeta$  on gravitational lensing may be detected in the future. It would be of interest to study the Hawking radiation and quasinormal modes of the slowly-rotating black hole in dynamical Chern-Simons modified gravity. Work in this direction will be reported in the future.

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